The ETH Math Youth Academy

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The ETH Math Youth Academy, recently established in Fall 2015, has been especially designed for Kurzzeitgymnasium students who want to engage in creative thinking and to delve deeper into exciting mathematics. The regular weekly classes are in the form of mini-courses on various extracurricular topics. The purpose of the present article is to provide details about the project and to invite teachers to collaborate.

1 SwissMAP and a mathematical "outreach" position at ETHZ

SwissMAP[1] is a large–scale, long–term project, funded by a grant from the Swiss National Science foundation, whose primary goal is to promote advanced–level research in mathematics and theoretical physics.

The project is so large that it also includes mathematical "outreach" activities for students in school. SwissMAP–funded outreach was first introduced in the French–speaking part of Switzerland, and has now further expanded to the German–speaking part. Namely, a new particular postdoc position at ETH Zürich has been recently created, with a focus on *designing* and *implementing* mathematical outreach activities for high school students. As I am holding this position since September 2015, I would like to describe the program that I have designed and launched. It is intended to be a long–term project and to grow over the upcoming years.

2 The weekly classes

The main activity of the ETH Math Youth Academy consists of regular weekly classes on topics outside of the standard gymnasium or university mathematics curriculum. There is an abundance of interesting, inspiring, and conceptual mathematics that is completely accessible to high school students and can be readily appreciated by them. Concrete examples will be given in Section 5. There is neither cost nor credit for these classes — they are meant for the personal enrichment and academic satisfaction of the students.

The course is intended for students in Kurzzeitgymnasium (above the age of 14). There is a "main" course for students in their early years of the gymnasium or students new to extracurricular mathematics, and a separate "advanced" course for the more experienced ones. The style and teaching philosophy is the same for both classes, but the topics in the advanced class are more difficult.

The main course meets on Wednesdays from 17:15 to 19:00 at ETH Zentrum. The schedule of the advanced class is coordinated with the schedules of the students attending and may vary from semester to semester; in Spring 2016, it takes place on Mondays from 17:15 to 19:00. Classes do not take place during school holidays.

[1] `http://www.nccr-swissmap.ch`

The style of teaching is an interactive lecture format: I present material on the board, while continuously asking students for input and suggestions. I attempt to guide them to brainstorm ideas, build arguments, or, ideally, arrive at a solution. I spell out and write on the board the proofs with complete rigor: one of the goals of the course is that students become comfortable with the writing of rigorous and logically correct proofs.

2.1 Content

The classes are organized by topic or technique in the form of mini–courses, each lasting for 3 to 6 sessions. The focus is on elementary and yet nontrivial conceptual mathematics, whose aesthetics can be readily appreciated by the students. Some of the great ideas of mathematics manifest themselves already in settings which require only gymnasium–level background to understand. Moreover, exposure to such outside–of–the–box mathematical thinking is beneficial not only for the students who continue with mathematics in the university: the development of creative thinking and logical argumentation skills are advantageous in fact for any profession.

The classes take a middle–ground between what is regarded as "olympiad preparation/problem–solving" and theoretical mathematical studies. They are neither isolated problem–solving sessions, nor do they cover standard topics from the first years in a university. Preparation for mathematical olympiads is certainly a biproduct, but not the main goal in itself. Due to the systematic exposure to proofs, students will be better prepared for theoretical endeavors in the university.

2.2 Target audience and prerequisites

Geographically, any student willing to travel to ETH Zentrum is welcome. As of March 2016, regularly attending students come from the following schools: Literargymnasium Rämibühl, Realgymnasium Rämibühl, MNG Rämibühl, KS Freudenberg, KS Zürcher Oberland, KS Solothurn.

In terms of prerequisites, *any* student interested in pursuing extracurricular mathematics is very welcome to attend the classes! There is absolutely no specific background necessary — except, of course, comfort with the foundational material studied in school. In fact, students from the early years of the gymnasium are particularly encouraged to participate, since we can build more background over the years and delve into more advanced and sophisticated topics in the upcoming years.

I do not like to say that this is for the "gifted" or "talented" students, since it may often be too difficult or even impossible to judge correctly who is gifted at this age and who is not. In fact, those students are often modest and would not describe themselves as "gifted", so they might be scared and decide that this is not for them.

Instead, this opportunity is aimed for the **interested** students. Any student motivated to spend these two hours after school doing extra mathematics is welcome. In fact, I encourage students in doubt to give it a try and then decide if they would like to continue or not. This is what happened when the project was launched: about 40 students from 6 schools attended the first two classes, and then about 15 keen ones continued to attend regularly. If more students from more schools hear about the project, they can check it out and make an informed decision if it is a good fit for them.

2.3 Model

These "math circle" classes model well–established traditions in Eastern Europe. For example, in my high school in Bulgaria, there were in fact separate such weekly classes for the students in each year of the gymnasium. Math circles are popular in the United States — the Berkely math circle is a prominent example which illustrates the scale that such a project can reach: `http://mathcircle.berkeley.edu`.

According to [3], Switzerland (along with Liechtenstein) is best in Europe and 9-th in the world in terms of quality of the general–level mathematical education in school. However, the ETH Math Youth Academy is an opportunity for the students who want to engage further with mathematics to an *extracurricular* level. A student who enjoys music can play in an orchestra; a student who enjoys mathematics can take advantage of this new educational opportunity.

3 Public talks

Parallel to the weekly classes, I have launched an initiative for giving public talks at various high schools in Switzerland, emphasizing the aesthetic side of mathematics. I would love to communicate some of the most beautiful pieces of mathematics and to present them in an elementary and accessible way. These talks are meant to be short, up to 30 minutes (possibly less) and could fit into the school schedules during the lunch break, for example. They are designed for a *broad general audience* and are intended to be interesting and inspiring even for students whose favorite subject in school is not mathematics. The participation of students in a special school event with a guest speaker from ETH Zürich would add to the weight and popularity of mathematics as a school subject.

The first such talk that I gave took place at the Literargymnasium Rämibühl, thanks to the initiative of the math teacher Mrs. Margot Sigrist–Constantin, and the second one — at KS Baden, thanks to the enthusiasm of Mr. Armin Barth. They are available online

- Mathematical induction, `https://www.youtube.com/watch?v=ZgCHTAmwtm4`;
- Mathematical games, linked from `https://www.math.ethz.ch/eth-math-youth-academy`,

and I would encourage the readers to watch them for a concrete idea of how they work.

I have an abundance of further topics in mind, and I am looking forward to giving more talks. As an example, some of the problems discussed in Section 5 below would be appropriate as the content of a short public talk. As another example, in a talk on "Mathematical colorings," I would start with the classical problem that if the bottom left and top right square of a chessboard have been removed, then the remaining squares cannot be tiled by 2×1 domino pieces (this is because in the standard coloring of the chessboard, each domino piece covers one white and one black square, while there are 30 black and 32 white squares remaining), and then discuss a second problem that involves a slightly fancier coloring. Further possible topics include: invariants, Pythagorean triples, number bases, prime numbers, Fibonacci numbers.

4 Your collaboration

The high participation rate and the success of the ETH Math Youth Academy so far have been due to the tremendous support that I had received from mathematics teachers in the months when I was informing the community about the project to be launched. My goals for the first year have been surpassed, and I am extremely grateful to all teachers who have contributed.

I strive for the project to expand significantly over its second year and to continue growing in the upcoming years. Therefore, I would like to invite teachers to promote its further development by

- arranging that I give a public talk at their school;
- informing their students about the weekly classes and encouraging them to participate;

- forwarding this announcement to other teachers or school administrators.

The ETH Math Youth Academy website is available at

`https://www.math.ethz.ch/eth-math-youth-academy`.

5 A sample class – the Extreme principle

This is a sample session from the main class. The "Extreme principle" refers to the general technique of picking an extreme element — smallest or largest — in a given setting or configuration. This could be a distance, an angle, an area (in geometry), a number that counts an appropriately–chosen quantity (in combinatorics), or a smallest solution of an equation (in number theory). Once such an extreme element has been singled out, just as a crack in the rock that a rockclimber grabs, it serves as a starting point in approaching the problem, and allows us to make deductions and build arguments.

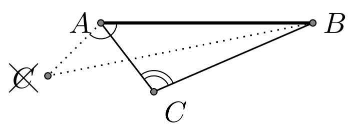
My view is that the Extreme Principle is most intuitively and vividly illustrated first in a *geometric* context, so the first few sessions in which I introduced it contained examples from geometry. Later on, thinking about an extreme element in combinatorial or number–theoretic settings will be seen by the students as a more natural approach.

For some of the problems below, I present just a solution, with only some hints about the guidance that I was giving, and omitting from the exposition the discussion that had taken place in class. For other problems, inspired by the style of [2], I also include samples from the classroom discussion. All of the problems presented here can be found in [1] or [4].

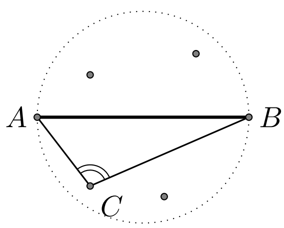
Problem 1. Finitely many points are given in the plane. It is known that the distance between any two of them is at most 1, and any triangle formed by three of these points has an obtuse angle. Prove that all the points are contained in some disc of diameter at most 1.

Solution. Since there are finitely many points given, there are also finitely many distances between pairs of points. Therefore, there is a *largest* distance: let A and B be two of the points, for which the distance AB is largest. If there are several such pairs, just take any one of them.

Now, by using the special choice of the points A and B, we can make a statement about the location of any *other* point C among the given ones. Where can C possibly be located? The triangle ΔABC has an obtuse angle — where can that obtuse angle be? It cannot be that $\sphericalangle BAC > 90°$, since then $CB > AB$, contradicting our choice of a *largest* distance AB. Therefore, we know that specifically, $\sphericalangle ACB > 90°$.

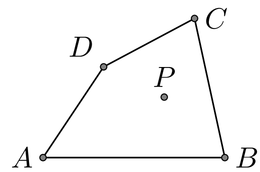


But this implies that any other point C is within the disc with diameter AB (recall that points on the circle with diameter AB satisfy $\sphericalangle ACB = 90°$, exterior points C satisfy $\sphericalangle ACB < 90°$, and interior points C satisfy $\sphericalangle ACB > 90°$). Since $AB \leq 1$, the disc with diameter AB satisfies the requirements.

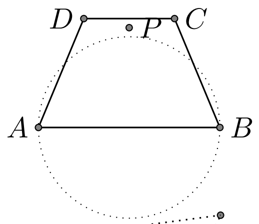


Problem 2. Let $ABCD$ be a convex quadrilateral. Prove that the discs with diameters AB, BC, CD, DA cover the whole interior of $ABCD$.

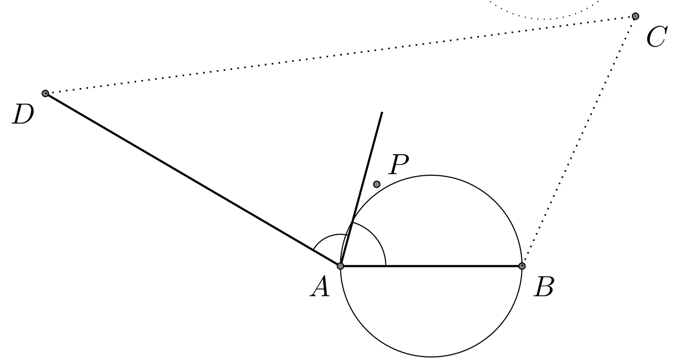
Discussion. Let P be an interior point. We have to prove that it belongs to one of these 4 discs. Which one? How would we select out a special side of the quadrilateral, to be sure that the respective disc contains the point P?



— Take the *longest* side of $ABCD$.
— But let's look at the following drawing. Here, this largest disc doesn't cover P. Moreover, it would be strange if this works, since then a *single* disc would cover the entire quadrilateral. You expect the target side to depend also on the point P.

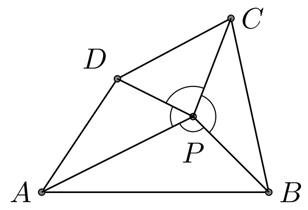


— OK, then, take the side *closest* to P.
— This is a good guess; however, it could be that P is closest to the side AB, but still outside of the disc with diameter AB, as in the following picture.



— It is not the distance from a point to a side that tells you if the point belongs to the disc with diameter this side. Think about the previous problem. What does it mean quantitatively that P belongs to the disc with diameter, for example, AB?
– It means that $\sphericalangle APB \geq 90°$.

— Excellent! Now, how do we select out the required side, after all?
— One of the four angles, $\sphericalangle APB, \sphericalangle BPC, \sphericalangle CPD, \sphericalangle DPA$ must be at least $90°$.
— Why?
— They add up to $360°$.
— Exactly: they add up to $360°$, so the *largest* angle among them must be at least $90°$: if the largest angle were less than $90°$, their sum would be less than $4 \times 90° = 360°$.



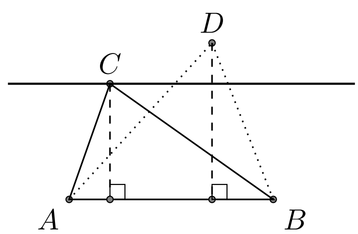
Problem 3. Finitely many points are given in the plane. It is known that the area of the triangle formed by any three of them is at most 1. Prove that all the points can be placed within a triangle of area at most 4.

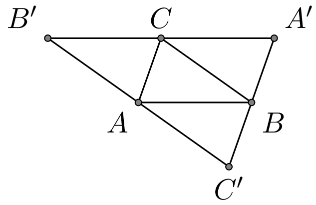
Solution. What we are given and what we have to prove concern areas. We take a triangle ΔABC with vertices among the given points, whose area is *largest*. We will use it to

- constrain the possible positions of all the *other* points;
- construct the required triangle of area at most 4.

How does the choice of ΔABC with largest area constrain the locations of the other points? Can you describe some region in the plane that for sure must contain all the other points?

If we a draw a line through C parallel to AB, then all the other points must lie in the same half–plane determined by this line, as the points A, B — else, we would have a triangle ΔABD of area larger than that of ΔABC.



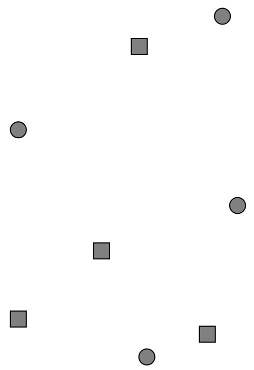


Therefore, if we build a triangle $\Delta A'B'C'$ by drawing similar lines through B and C, all of our points will lie within $\Delta A'B'C'$. Notice the three parallelograms in the figure. So, the area of $\Delta A'B'C'$ is

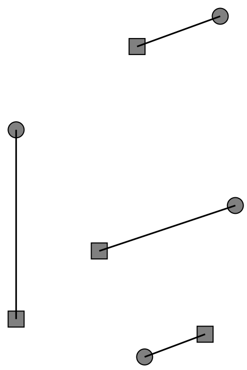
$$S_{A'B'C'} = 4S_{ABC} \leq 4.$$

Problem 4. There are n farms and n wells in the plane, so that no three of the $2n$ points that they represent lie on a line. Prove that one can build n segments, each joining a farm and a well, such that no two of them intersect. In other words, one can match the farms and the wells via non–crossing straight roads.

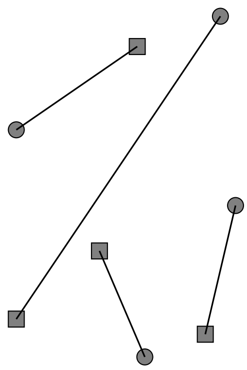
Discussion. Let's mark farms by squares, and wells by circles.



—How would you connect the farms and wells in this example?

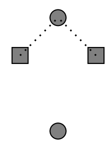


A student suggests this matching.

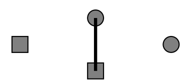


— By the way. there are many solutions possible. Here is another, less "natural" one.

—This example could give us some feeling for the problem, but now we need to give a general proof. How would you construct a non–crossing matching?



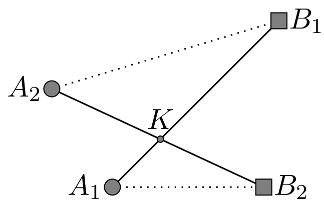
— Connect each farm to the *closest* well.
— This is a good start. A small problem is that some farm might have two equally close wells. If you say that you connect each farm to *a* closest well, then a problem is that some well might be the one closest to each of two farms.



— OK, then, pick one particular farm, join it to a well closest to it, isolate the pair, and continue.
— Good idea. However, if we join a pair in this manner, we might get stuck and might not be able to continue, even if on the first step we had chosen a smallest distance between a farm and a well whatsoever.
— You are certainly on the right track, thinking about *short* segments and connecting points which are *close* to each other. It would be very unnatural to draw "long" segments, since you would expect that they would block possibilities to match the remaining points. However, you need to give a construction for the figure as a whole, all at once, and not give a "greedy" preference for a point or a pair that you join first, since it may then block other pairs.

There are finitely many possible matchings between farms and wells whatsoever (how many? exactly $n!$, in fact). Among all such matchings, with crossings or not, pick one for which the *sum* of all the drawn segments is *smallest*. Let's prove that for this particular (extreme) matching, there are in fact *no* intersections.

Suppose that A_1B_1 and A_2B_2 are two segments in this extreme matching, each joining a farm and a well, which intersect at a point K. We have to obtain a contradiction. What is going wrong here?



— It seems that

$$A_1B_2 + A_2B_1 < A_1B_1 + A_2B_2. \tag{1}$$

— Great! If we prove (1), we obtain the desired contradiction: if instead of matching A_1 with B_1 and A_2 with B_2, we match A_1 with B_2 and A_2 with B_1, keeping the rest of the matching unchanged, we obtain a matching with overall *smaller* total length! Now, why does (1) hold?
— Because the sum of the lengths of the diagonals is larger than the sum of the lengths of the opposite sides.
— Well, this is *restating* the claim in words, but is not a proof. You have to use the existence of this point K, so think about bringing it into the game.
— By the triangle inequality,

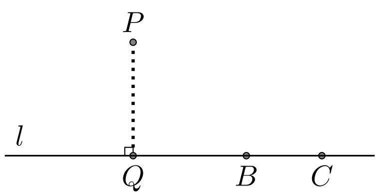
$$A_1B_2 < A_1K + KB_2$$
$$A_2B_1 < A_2K + KB_1.$$

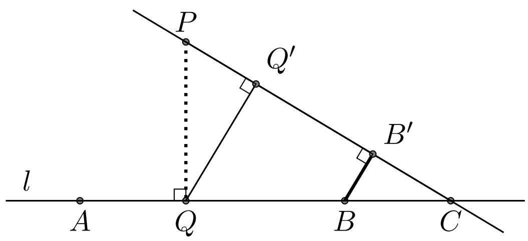
Adding these two inequalities and combining the terms $A_1K + KB_1 = A_1B_1$ and $A_2K + KB_2 = A_2B_2$, we deduce (1).

Problem 5. (Sylvester–Gallai theorem) Finitely many points are given in the plane. The line through any two of them passes through a third. Prove that all the points lie on a line.

Solution. Suppose that not all of the points lie on a line. Since the given points are finitely many, the number of lines joining pairs of them is also finite.

Among all the nonzero distances between a point among the ones given and a line joining two of the points given, take a *shortest* one. (Why does at least one nonzero distance exist?) Say the distance PQ between the point P and the line l is shortest. The line l joins two of the points given, and so, by assumption, it contains in fact at least *three* of them. At least two of them, say B and C, will be on the same side of Q (possibly, one coincides with Q). Without loss of generality, B is between Q and C.





But now, look at the distance between the point B and the line joining P and C:

$$0 < BB' \leq QQ' < PQ.$$

So, BB' is a nonzero distance between a point among the ones given and a line determined by two of them, which is smaller than the distance between P and l, contradicting the choice of P and l!

This problem is a perfect example of elementary and yet nontrivial mathematics. It has a rich history: the question was raised in 1893 and was not solved in fact until about 40 years later. This is a beautiful example of inspiring and *vivid* mathematics that is perfectly accessible and readily appreciated by interested high school students.

I had spent a total of 3 weeks on geometric illustrations in the spirit of the above problems, including a number of further problems solved by the subtle idea from Problem 5 above, so that students can become comfortable with it. After each class, I compose a list of problems as optional homework, for the students who can afford the time to try them before the next class.

Next, I devoted a session to the Extreme Principle in *non–geometric* settings. It will prominently reappear later in number–theoretic settings.

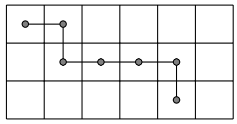
Problem 6. A positive integer has been written down in each square of an infinite square grid, so that each entry is the average of its four neighbors (sharing a side). Prove that all numbers written down are equal.

Solution. Any set of positive integers contains a smallest element. Let a be the *smallest* number written anywhere in the grid.

Consider any cell that contains the number a, and let $b_1, ..., b_4$ be its four neighbors. Since a is the average of the b's *and* $a \le b_i$ for each i, then $a = b_i$ for each i.

...	b_1	...
b_2	a	b_4
...	b_3	...

So, any square adjacent to one containing a also contains a. Any cell in the grid can be reached from a cell containing a by a zig–zag path consisting of horizontal or vertical segments. Repeatedly applying our conclusion along such a path, we deduce that the number in each cell of the grid is a.



Problem 7. Each member of the Parliament has at most 3 enemies. Prove that the members can be split into two Houses, so that everyone has at most one enemy in their own House.

Solution. Among all possible splittings of the members into two Houses, take one for which the total number of pairs of enemies within the same House is *smallest*. We claim that this (extreme) splitting satisfies the required property. Suppose, for the sake of contradiction, that a member X of House A has at least 2 enemies in House A. Therefore, X has at most 1 enemy in House B. Move X to House B. The number of enemy pairs in House A will decrease by at least 2, while the number of enemy pairs in House B will increase by at most 1. Hence, the total number of enemy pairs within a house will *decrease* — a contradiction. Compare with Problem 4.

Notice the modest background required for any of the above problems. Any high school student who likes mathematics and is comfortable with the school material would be able to participate in, understand, and appreciate these classes.

In the first months, the emphasis in the main class has been on combinatorics (elementary graph theory, invariants, mathematical colorings, mathematical games, the Pigeonhole principle). Towards the end of the academic year, the shift will move to number theory. I will start developing it completely from scratch, by introducing systematically basic congruences, Euclid's algorithm for division with remainder, properties of the prime numbers, existence of infinitely many primes, unique factorization, Fermat's Little Theorem. These sessions will be more theoretical in nature and will be building upon one another to a higher degree. However, the theory will be continuously illustrated by numerous concrete examples and applications, such as cryptography and Diophantine equations (describing all Pythagorean triples, for example).

References

[1] A. Engel, *Problem–solving strategies*, 1998, Springer–Verlag NY.

[2] D. Fomin, S. Genkin, I. Itenberg, *Mathematical circles (Russian experience)*, 1996, AMS.

[3] `http://www.oecd.org/pisa/keyfindings/pisa-2012-results.htm`

[4] `www.problems.ru` (in Russian)